

Quantum energy-mass spectrum of Yang-Mills bosons

Alexander Dynin

Department of Mathematics, Ohio State University

Columbus, OH 43210, USA, dynin@math.ohio-state.edu

Abstract

A non-perturbative quantization of the Yang-Mills energy-mass functional with a compact semi-simple gauge group entails an infinite discrete energy-mass spectrum of gauge bosons. The bosonic spectrum is bounded from below, and has a positive mass gap due to the quartic self-interaction term of pure Yang-Mills Lagrangian (with no Higgs term involved). This quantization is based on infinite-dimensional analysis in Kree nuclear triple of sesqui-holomorphic functionals of initial data for the the non-linear classical Yang-Mills equations in the temporal gauge.

Key words: Yang-Mills theory; nuclear triples; energy-mass functional; bosonic spectrum; quantum Yang-Mills mass gap.

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1 Introduction

1.1 Physical background

A mass gap for the subnuclear weak and strong forces is suggested by experiments in accordance with Yukawa principle: *A limited force range suggests a massive carrier.*

As this paper shows, a positive mass gap for a *pure* quantum relativistic Yang-Mills theory (with any semi-simple gauge group) can be mathematically deduced from the *quartic* self-interaction term in the pure relativistic Yang-Mills Lagrangian with no Higgs term.

The energy-mass time component P^0 of the classical Noether energy-momentum relativistic vector P^μ is not a Poincare scalar. I work only in the *rest Lorentz frames* where the energy-momentum vector is a direction vector of the time axis. The Yang-Mills energy-momentum 4-vector is time-like (see GLASSEY-STRAUSS [9]).

Any Poincare frame is relativistically equivalent to a Yang-Mills rest Lorentz frame, and any Yang-Mills gauge is equivalent to the temporal gauge (see, e.g, FADDEEV-SLAVNOV [8, Chapter III, Section 2]. Since the quantization is invariant with respect to the residual Poincare and gauge symmetries, the bosonic spectrum and its spectral gap are *relative* Poincare and gauge quantum invariants.

The paper is a formulation and a proof of the spectral Theorem 4.1. This is done in four steps:

- A** In the temporal gauge, Yang-Mills fields (i.e. solutions of *relativistic* Yang-Mills equations) are in one-one correspondence with their constrained Cauchy data (see, e.g. GOGANOV-KAPITANSKII [10]).

This parametrization of the classical Yang-Mills fields is advantageous in three ways:

- Cauchy data carry a *positive definite* scalar product.
- The non-linear constraint equation for Cauchy data is *elliptic*.
- Yang-Mills energy-mass functional of Cauchy data is *not* a relativistic conformal invariant.

The elliptic equation is solved via a gauge version of classical Helmholtz decomposition of vector fields. The solution provides a *linearization of non-linear constraint manifold* of Cauchy data.

- B** In the line of I. Segal non-linear quantization program (see, SEGAL [16]) along with the quantum postulate of BOGOLIUBOV-SHIRKOV [3, Chapter II]) I quantize the rest energy-mass functional of Cauchy data as a tame operator in a Kree nuclear triple of [11], [12] (see also MEYER[13]). Note that according to the quantum postulate no operators fields are involved. The quantization is chosen to be *anti-normal* (aka anti-Wick, Berezin, or diagonal quantization).
- C** I extend AGARVAL-WOLF [1] symbolic calculus to show that the Weyl symbol of the anti-normal energy-mass operator contains a *positive*

quadratic mass term which is absent in the classical energy-mass functional.¹

- D** It follows that the anti-normal energy-mass operator dominates a shift of the number operator. Splitting off the invariant spaces that are irreducible under bosonic permutations, I show that the corresponding bosonic spectrum is *infinite and discrete*.

All new defined terms in the text are introduced via emphasizing in italics. The beginning and the end of a proof are marked by \triangleright and \triangleleft .

2 Yang-Mills energy-mass functional

2.1 Gauge groups

The *global gauge group* \mathbb{G} of a Yang-Mills theory is a connected semi-simple compact Lie group with the Lie algebra $\text{Ad}(\mathbb{G})$.

The notation $\text{Ad}(\mathbb{G})$ indicates that the Lie algebra carries the adjoint representation $\text{Ad}(g)X = gXg^{-1}$, $g \in \mathbb{G}$, $a \in \text{Ad}(\mathbb{G})$, of the group \mathbb{G} and the corresponding self-representation $\text{ad}(X)Y = [X, Y]$, $X, Y \in \text{Ad}(\mathbb{G})$. Then $\text{Ad}(\mathbb{G})$ is identified with a Lie algebra of skew-symmetric matrices and the matrix commutator as Lie bracket with the *positive definite* Ad-invariant scalar product

$$X \cdot Y \equiv \text{Trace}(X^T Y), \quad (2.1)$$

where $X^T = -X$ denotes the matrix transposition (see, e.g. ZHELOBENKO [18, section 95]).

Let the Minkowski space \mathbb{M} be oriented and time oriented with the Minkowski metric signature $(+, -, -, -)$. In a Minkowski coordinate systems x^μ , $\mu = 0, 1, 2, 3$, the metric tensor is diagonal. In the natural unit system, the time coordinate $x^0 = t$. Thus $(x^\mu) = (t, x^i)$, $i = 1, 2, 3$.

The *local gauge group* \mathcal{G} is the group of infinitely differentiable \mathbb{G} -valued functions $g(x)$ on \mathbb{M} with the pointwise group multiplication. The *local gauge Lie algebra* $\text{Ad}(\mathcal{G})$ consists of infinitely differentiable $\text{Ad}(\mathbb{G})$ -valued functions on \mathbb{M} with the pointwise Lie bracket.

\mathcal{G} acts via the pointwise adjoint action on $\text{Ad}(\mathcal{G})$ and correspondingly on \mathcal{A} , the real vector space of *gauge fields* $A = A_\mu(x) \in \text{Ad}(\mathcal{G})$.

¹For other applications of the symbolic calculus to quantum dynamics see DYNIN [7].

Gauge fields A define the *covariant partial derivatives*

$$\partial_{A\mu}X \equiv \partial_\mu X - \text{ad}(A_\mu)X, \quad X \in \text{Ad}(\mathcal{G}). \quad (2.2)$$

Any $g \in \mathcal{G}$ defines the affine *gauge transformation*

$$A_\mu \mapsto A_\mu^g \equiv \text{Ad}(g)A_\mu - (\partial_\mu g)g^{-1}, \quad A \in \mathcal{A}, \quad (2.3)$$

so that $A^{g_1}A^{g_2} = A^{g_1g_2}$.

2.2 Yang-Mills fields

Yang-Mills *curvature tensor* $F(A)$ is the antisymmetric tensor

$$F(A)_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu]. \quad (2.4)$$

The curvature is gauge covariant:

$$\partial_{A\mu}\text{Ad}(g) = \text{Ad}(g)\partial_{A\mu}, \quad \text{Ad}(g)F(A) = F(A^g). \quad (2.5)$$

The *Yang-Mills Lagrangian* (with the coupling constant set to 1)

$$L = -(1/4)F(A)^{\mu\nu} \cdot F(A)_{\mu\nu} \quad (2.6)$$

is invariant under gauge transformations.

The corresponding Euler-Lagrange equation is a 2nd order non-linear partial differential equation $\partial_{A\mu}F(A)^{\mu\nu} = 0$, called the *Yang-Mills equation*

$$\partial_\mu F^{\mu\nu} + [A_\mu, F^{\mu\nu}] = 0. \quad (2.7)$$

The solutions A are *Yang-Mills fields*. They form the *on-shell space* \mathcal{M} of the classical Yang-Mills theory.

From now on we assume that all space derivatives of gauge fields $A = A(t, x^k)$ vanish faster than any power of $x^k x_k$ as $x^k x_k \rightarrow \infty$, uniformly with respect to bounded t . (This condition does not depend on a Lorentz coordinate system.) Let $\text{Ad}\mathcal{G}$ denote the local Lie algebra of such gauge fields and \mathcal{G} denote the corresponding infinite dimensional local Lie group.

In a Lorentz coordinate system we have the following matrix-valued time-dependent fields on \mathbb{R}^3 :

$$\text{Gauged electric vector field } E(A) \equiv (F_{01}, F_{02}, F_{03}),$$

Gauged magnetic pseudo vector field $B(A) \equiv (F_{23}, F_{31}, F_{12})$.

Now the (non-trivial) energy-mass conservation law is that the time component

$$P^0(A) \equiv \int d^3x (1/2)(E^i \cdot E_i + B^i \cdot B_i) \quad (2.8)$$

of the relativistic Noether energy-momentum vector is constant on-shell. Appropriately, $P^0(A)$ has the mass dimension.

At the same time, by Glassey-Strauss Theorem [9], the *energy-mass density* $(1/2)(E^i \cdot E_i + B^i \cdot B_i)$ scatters asymptotically along the light cone as $t \rightarrow \infty$. This is a mathematical reformulation of the physicists assertion that Yang-Mills fields propagate with the light velocity.

2.3 First order formalism

Rewrite the 2nd order Yang-Mills equations (2.7) in the temporal gauge $A_0(t, x^k) = 0$ as the 1st order systems of the *evolution equations* for the time-dependent $A_j(t, x^k), E_j(t, x^k)$ on \mathbb{R}^3 as

$$\partial_t A_k = E_k, \quad \partial_t E_k = \partial_j F_k^j - [A_j, F_k^j], \quad F_k^j = \partial^j A_k - \partial_k A^j - [A^j, A_k]. \quad (2.9)$$

and the *constraint equations*

$$[A^k, E_k] = \partial^k E_k, \quad \text{i.e.} \quad \partial_{k,A} E_k = 0 \quad (2.10)$$

By GOGANOV-KAPITANSKII [10], the evolution system is a semilinear first order partial differential system with *finite speed propagation* of the initial data, and the Cauchy problem for it with initial data at $t = 0$

$$a(x_k) \equiv A(0, x_k), \quad e(x_k) \equiv E(0, x_k) \quad (2.11)$$

is *globally and uniquely solvable on the whole Minkowski space* \mathbb{M} .

Actually, this was proved in [10] without any restriction on Cauchy data at the infinity.

If the constraint equations are satisfied at $t = 0$, then, in view of the evolution system, they are satisfied for all t automatically. Thus the *1st order evolution system along with the constraint equations for Cauchy data is equivalent to the 2nd order Yang-Mills system*. Moreover the constraint equations are invariant under *time independent* gauge transformations. As the bottom line, we have

Proposition 2.1 *In the temporal gauge Yang-Mills fields A are in one-one correspondence with their gauge transversal Cauchy data (a, e) satisfying the equation $\partial_a e = 0$.*

Let $\mathcal{A}^0 = \mathcal{A}^0(\mathbb{R}^3)$ denote the real \mathcal{L}^2 -space of Cauchy gauge vector fields a on \mathbb{R}^3 . The associated Sobolev-Hilbert spaces (see, e.g. SHUBIN [17, Section 25]) are denoted \mathcal{A}^s , $s \in \mathbb{R}$. The intersection $\mathcal{A}_0^\infty \equiv \bigcap_s \mathcal{A}^s$ with the projective limit topology is a nuclear real Frechet space of smooth a . The union $\mathcal{A}^{-\text{infty}} \equiv \bigcup_s \mathcal{A}^{-s}$ with the inductive limit topology is its dualspace of \mathcal{A}_0^∞ .

Let \mathcal{G}^s , $s > 3/2$, be the infinite dimensional Frechet Lie groups with the Lie algebras \mathcal{A}^s , $s > 3/2$.

The intersection $\mathcal{G}^\infty \equiv \bigcap_s \mathcal{G}^s$ is an infinite dimensional Lie group with the nuclear Lie algebra \mathcal{A}^∞ . The local gauge transformations a^g by $g \in \mathcal{G}^\infty$ define continuous left action $\mathcal{G}^\infty \times \mathcal{A}^s \rightarrow \mathcal{A}^{s-1}$.

Local gauge transformations

$$a_k^g = \text{Ad}(g)a_k - (\partial_k g)g^{-1}, \quad g \in \mathcal{G}^\infty, \quad a \in \mathcal{A}^s, \quad (2.12)$$

define continuous left action of \mathcal{G}^s on \mathcal{A}^s .

The Sobolev-Hilbert spaces \mathcal{S}^s of smooth Cauchy gauge electric fields e on \mathbb{R}^3 with the corresponding action e^s of the local gauge group \mathcal{G}^∞ are defined the same way.

2.4 Gauged vector calculus

Let \mathcal{U}^s denote the Sobolev-Hilbert spaces $\text{Ad}(\mathbb{G})$ -valued functions u on \mathbb{R}^3 .

Consider the continuous vector calculus operators gauged by $a \in \mathcal{A}^\infty$

Gauged gradient

$$\text{grad}_a : \mathcal{U}^s \rightarrow \mathcal{S}^{s-1}, \quad \text{grad}_a u \equiv \partial_k u - [a_k, u], \quad (2.13)$$

Gauged divergence

$$\text{div}_a : \mathcal{S}^s \rightarrow \mathcal{U}^{s-1}, \quad \text{div}_a e \equiv \partial_k e_k - [a_k, e_k], \quad (2.14)$$

Gauged Laplacian

$$\Delta_a : \mathcal{U}^s \rightarrow \mathcal{U}^{s-2}, \quad \Delta_a \equiv \text{div}_a u \text{grad}_a u, \quad (2.15)$$

The 1st order partial differential operators $-\text{grad}_a$ and div_a are adjoint with respect to the \mathcal{L}^2 scalar product:

$$\langle -\text{grad}_a u \mid v \rangle = \langle u \mid \text{div}_a v \rangle. \quad (2.16)$$

The gauged Laplacian Δ_a is a 2nd order partial differential operator. Since its principal part is the usual Laplacian Δ , the operator Δ_a is elliptic.

Proposition 2.2 *The gauge Laplacian Δ_a is an invertible operator from \mathcal{U}^{s+2} onto \mathcal{U}^s for all $s \geq 0$.*

Lemma 2.1 $\Delta_a u = 0$, $u \in \mathcal{U}_0^1$, if and only if $u = 0$.

▷ $u \cdot [a, u] = -\text{Trace}(uau - uua) = 0$ so that

$$u \cdot \text{grad}_a u = u \cdot \text{grad} u = (1/2) \text{grad}(u \cdot u) = 0. \quad (2.17)$$

This shows that for $u \in \mathcal{U}_0^1$ we have $\text{grad}_a u = 0$ if and only if $u = 0$. ◁

Next, by the equality (2.16),

$$\langle \Delta_a u \mid u \rangle = \langle -\text{grad}_a u \mid \text{grad}_a u \rangle, \quad u \in \mathcal{U}_0^1. \quad (2.18)$$

Thus $\Delta_a u = 0$, $u \in \mathcal{U}_0^1$, if and only if $u = 0$. ◁

Both Laplacian Δ and gauge Laplacian Δ_a map \mathcal{U}^{s+2} into \mathcal{U}^s .

The Laplace operator is invertible from \mathcal{U}^{s+2} onto \mathcal{U}^s whatever $s \geq 0$ is. Since $\Delta - \Delta_a$ is a 1st order differential operator, the operator $\Delta_a : \mathcal{U}^{s+2} \rightarrow \mathcal{U}^s$ is a Fredholm operator of zero index. Then, by Lemma 2.1, the inverse $\Delta_a^{-1} : \mathcal{U}^s \rightarrow \mathcal{U}^{s+2}$ exists for all $s \geq 0$. ◁

Proposition 2.2 shows that the operator $\text{div}_a : \mathcal{U}^s \rightarrow \mathcal{U}^{s-1}$ is surjective and the operator $\text{grad}_a : \mathcal{U}^s \rightarrow \mathcal{U}^{s-1}$ is injective.

Let

$$\Pi_a \equiv \text{grad}_a \Delta_a^{-1} \text{div}_a \quad (2.19)$$

Both Π_a and $\mathbf{1} - \Pi_a$ are pseudodifferential operators of order 0, and, therefore are \mathcal{L}^2 -bounded.

By computation,

$$\Pi_a^\dagger = \Pi_a, \quad \Pi_a^2 = \Pi_a, \quad \Pi_a \text{grad}_a = \text{grad}_a, \quad \text{div}_a(\mathbf{1} - \Pi_a) = 0. \quad (2.20)$$

Therefore Π_a is an \mathcal{L}^2 -orthogonal projector of \mathcal{U}^s onto the space of gauge longitudinal vector fields, i.e. the range of the operator $\text{grad}_a : \mathcal{U}^{s+1} \rightarrow \mathcal{U}^s$; and the operator $\mathbf{1} - \Pi_a$ is an \mathcal{L}^2 bounded projector of \mathcal{U}^s onto the space of gauge transversal vector fields, i.e. the null space of the operator $\text{div}_a : \mathcal{U}^s \rightarrow \mathcal{U}^{s-1}$.

Now Proposition 2.1 implies

Proposition 2.3 *In the temporal gauge Yang-Mills fields A are in one-one correspondence with the vector bundle of the ranges of projectors Π_a over the base \mathcal{A}^∞*

2.5 Gelfand-Schwartz triple of constrained Cauchy data

Let $\mathcal{T}_a^\infty \subset \mathcal{S}^\infty$ denote the nuclear Frechet space of gauge transversal gauge electric vector fields $e_a \equiv e - \Pi_a(e)$, and \mathcal{T}^0 be its completion in \mathcal{S}^0 .

The family of orthogonal projectors $a \mapsto \Pi_a$ is a continuous mapping of \mathcal{A}^∞ to the algebra of bounded operators on \mathcal{S}^0 . Since for a sufficiently close to a_o the operators $1 - \Pi_a + \Pi_{a_o}$ are invertible and $\Pi_a \Pi_{a_o} = \Pi_a(1 - \Pi_a + \Pi_{a_o})\Pi_{a_o}$, the continuous mappings $\Pi_a \Pi_{a_o} : \Pi_{a_o}(\mathcal{S}^0) \rightarrow \Pi_a(\mathcal{S}^0)$ are invertible. Thus the vector bundle \mathcal{T}^0 of the gauge transversal spaces \mathcal{T}_a^0 is a locally trivial real vector bundle over \mathcal{A}^∞ .

Gauge invariance of the constraint manifold of Cauchy data under the (residual) gauge group implies the gauge covariance of projectors $1 - \Pi_a$, and so of the bundles. Since a Hilbert bundle structure group is smoothly contractible (see KUIPER [14]), the bundle \mathcal{T}^0 is isomorphic to the trivial gauge covariant Hilbert space bundle over its base: an isomorphism is defined by a smooth family of orthonormal bases of the bundle fibers. All such trivialisations intertwine with the action of the residual gauge group. They define linearly isomorphic global Hilbert coordinate charts on the constraint Cauchy data manifold $\mathcal{C}^0 \cong \mathcal{A}^0 \times \mathcal{T}_0^0$.

From the N -representation of nuclear Schwartz \mathcal{S} -spaces (cp., e.g. REED-SIMON [15, Theorem V.13] on the fibers, we get a locally trivial bundle of nuclear Schwartz \mathcal{S} -spaces of constrained e -fields over the same base along with the natural Gelfand nuclear triple of real topological vector spaces

$$\mathcal{C} : \mathcal{C}^\infty \equiv \mathcal{A}_0^\infty \times \mathcal{T}_0^\infty \subset \mathcal{C}^0 \equiv \mathcal{A}^0 \times \mathcal{T}^0 \subset \mathcal{C}^{-\infty} \equiv \mathcal{A}_0^{-\infty} \times \mathcal{T}_0^{-\infty}. \quad (2.21)$$

where \mathcal{C}^∞ is a nuclear Frechet space of smooth (a, e^o) , and $\mathcal{C}_0^{-\infty}$ is the dual of \mathcal{C}_0^∞ , with the duality defined by the inner product in \mathcal{C}^0 .

The assignment $(a, e^o) \mapsto z = (1/\sqrt{2})(a + ie^o)$ converts the real Gelfand triple (2.21) into a complex Gelfand triple, so that $\Re \mathcal{C} \equiv \mathcal{A}$ and $\Im \mathcal{C} \equiv \mathcal{T}_o$ are its real and imaginary parts.

The complex conjugation

$$z^* = (1/\sqrt{2})(a + ie^o)^* \equiv (1/\sqrt{2})(a - ie^o), \quad z \mapsto z^* : \mathcal{C} \rightarrow \mathcal{C}^{-\infty} \quad (2.22)$$

The (anti-linear on the left and linear on the right) Hermitian product is defined on \mathcal{C}^0 as

$$z^*z \equiv (1/2) \int d^3x (a \cdot a + e^o \cdot e^o)(x) \quad (2.23)$$

extended to the anti-duality between \mathcal{C}^∞ and $\mathcal{C}^{-\infty}$. Accordingly, *the notation z is reserved for the elements of the space \mathcal{C}^∞ , and the notation z^* for the elements of the space $\mathcal{C}^{-\infty}$.* The result is

Proposition 2.4 *The constraint Cauchy manifold is a complex nuclear Gelfand-Schwartz triple.*

3 Quantization

3.1 Review of KREE [11], [12]

The triple \mathcal{C} is a complex nuclear triple with the Hermitian conjugation $*$. By KREE [11] and [12] (also MEYER[13]), there is the associated Kree nuclear triple with the induced Hermitian conjugation $*$

$$\text{Exp}(\mathcal{C}^{-\infty}) \subset \mathcal{B}(\mathcal{C}^{-\infty}) \subset \text{Ent}(\mathcal{C}^\infty) \quad (3.1)$$

where

- $\text{Ent}(\mathcal{C}^\infty)$ is the complete nuclear space of all entire holomorphic functionals on \mathcal{C}^∞ with the topology of compact convergence.
- $\mathcal{B}(\mathcal{C}^{-\infty})$ is the Bargmann subspace of square integrable entire holomorphic functionals on $\mathcal{C}^{-\infty}$ with respect to the Gauss-Minlos probability measure γ . This is a complete Hilbert space identified with its $*$ -dual with by the Hermitian form

$$\langle \Psi | \Phi \rangle \equiv \int d\gamma \Psi^*(z) \Phi(z^*), \quad (3.2)$$

where the $*$ -dual $\Psi^*(z)$ is the complex conjugate of $\Psi(z^*)$.

- $\text{Exp}(\mathcal{C}^{-\infty})$ is the nuclear space of entire functionals of exponential type on $\mathcal{C}^{-\infty}$ identified with the $*$ -dual of $\text{Ent}(\mathcal{C}^\infty)^*$ via the $*$ -duality induced by (3.2).

- The *exponential states*, aka coherent states, $e^\zeta \in \text{Exp}(\mathcal{C}^{-\infty})$, $\zeta \in \mathcal{C}^\infty$,

$$e^\zeta(z^*) \equiv e^{z^*\zeta}, \quad z^* \in \mathcal{C}^{-\infty}, \quad \langle e^\zeta | e^w \rangle = e^{\zeta^*w}, \quad (3.3)$$

belong to $\text{Exp}(\mathcal{C}^{-\infty})$. They form the *continual exponential basis* of $\text{Ent}(\mathcal{C}^\infty)$, with the *continual coordinates*

$$\tilde{\Psi}(\zeta) \equiv \langle \Psi^* | e^\zeta \rangle, \quad \Psi \in \text{Ent}(\mathcal{C}^\infty). \quad (3.4)$$

This *complex Fourier transform* (aka Borel transform) is a topological *-automorphism of the Kree triple: For $\Psi \in \text{Ent}(\mathcal{C}^\infty)$, $\Phi \in \text{Exp}(\mathcal{C}^\infty)$

$$\langle \Psi^*(\zeta) | \Phi(\zeta^*) \rangle = \langle \Psi^*(z) | \Phi(z^*) \rangle. \quad (3.5)$$

By Grothendieck kernel theory, the nuclearity of the Kree triple implies that the locally convex vector spaces $\mathcal{O}(\cdot \rightarrow \cdot)$ of continuous linear operators are topologically isomorphic to the complete sesqui-linear tensor products (both spaces are endowed with the topology of compact uniform convergence).

$$\mathcal{O}(\text{Exp}(\mathcal{C}^{-\infty}) \rightarrow \text{Ent}(\mathcal{C}^\infty)) \simeq \text{Ent}(\mathcal{C}^\infty)^* \hat{\otimes} \text{Ent}(\mathcal{C}^\infty), \quad (3.6)$$

$$\mathcal{O}(\text{Exp}(\mathcal{C}^{-\infty}) \rightarrow \text{Exp}(\mathcal{C}^{-\infty})) \simeq \text{Ent}(\mathcal{C}^\infty)^* \hat{\otimes} \text{Exp}(\mathcal{C}^{-\infty}), \quad (3.7)$$

$$\mathcal{O}(\text{Ent}(\mathcal{C}^\infty) \rightarrow \text{Exp}(\mathcal{C}^{-\infty})) \simeq \text{Exp}(\mathcal{C}^{-\infty})^* \hat{\otimes} \text{Exp}(\mathcal{C}^{-\infty}) \quad (3.8)$$

where the operators are *tame* in the case of (3.7) and *smoothing* in the case of (3.8).

The formulas present the one-to-one correspondence between operators and the sesqui-holomorphic kernels of their matrix elements.

The nuclear Gelfand-Schwartz triple of the sesqui-Hermitian direct products

$$(\mathcal{C}^{-\infty})^* \times \mathcal{C}^{-\infty} \subset (\mathcal{C}^0)^* \times \mathcal{C}^0 \subset (\mathcal{C}^\infty)^* \times \mathcal{C}^\infty \quad (3.9)$$

carries the Hermitian conjugation

$$(z^*, w)^* \equiv (w^*, z) \quad (3.10)$$

The associated Kree triple of sesqui-holomorphic kernels consists of

$$\text{Exp}((\mathcal{C}^{-\infty})^* \times \mathcal{C}^{-\infty}) \simeq \text{Exp}(\mathcal{C}^{-\infty})^* \hat{\otimes} \text{Exp}(\mathcal{C}^{-\infty}), \quad (3.11)$$

$$\mathcal{B}((\mathcal{C}^{-\infty})^* \times \mathcal{C}^{-\infty}) \simeq \mathcal{B}(\mathcal{C}^{-\infty})^* \hat{\otimes} \mathcal{B}(\mathcal{C}^{-\infty}), \quad (3.12)$$

$$\text{Ent}((\mathcal{C}^\infty)^* \times \mathcal{C}^\infty) \simeq \text{Ent}(\mathcal{C}^\infty)^* \hat{\otimes} \text{Ent}(\mathcal{C}^\infty), \quad (3.13)$$

where $\mathcal{B}((\mathcal{C}^{-\infty})^* \times \mathcal{C}^{-\infty})$ is the Bargmann-Hilbert space with respect to the direct product of the probability Radon measures on $(\mathcal{C}^{-\infty})^* \times \mathcal{C}^{-\infty}$.

The corresponding exponential functionals are

$$e^{(\zeta^*, \eta)}((z^*, w)^*) = e^{w^*\eta + \zeta^*z}. \quad (3.14)$$

3.2 Operator symbols

Kree triple (3.1) carries the bosonic canonical linear representation of the triple \mathcal{C} by continuous linear transformations of $\zeta \in \text{Exp}(\mathcal{C}^\infty)$ and $\zeta^* \in \text{Ent}(\mathcal{C}^\infty)$ into the adjoint operators of *creation and annihilation* continuous operators of multiplication and directional differentiation

$$\hat{\zeta} : \text{Exp}(\mathcal{C}^{-\infty}) \rightarrow \text{Exp}(\mathcal{C}^{-\infty}), \quad \hat{\zeta} \Psi(z^*) \equiv (z^* \zeta) \Psi(z^*), \quad (3.15)$$

$$\widehat{\zeta^*} : \text{Ent}(\mathcal{C}^\infty) \rightarrow \text{Ent}(\mathcal{C}^\infty), \quad \widehat{\zeta^*} \Psi(z) \equiv (\zeta^* z) \Psi(z), \quad (3.16)$$

$$\widehat{\zeta^*}^\dagger : \text{Exp}(\mathcal{C}^{-\infty}) \rightarrow \text{Exp}(\mathcal{C}^{-\infty}), \quad \widehat{\zeta^*}^\dagger \Psi(\zeta^*) \equiv \partial_{\zeta^*} \Psi(\zeta^*), \quad (3.17)$$

$$\hat{\zeta}^\dagger : \text{Ent}(\mathcal{C}^\infty) \rightarrow \text{Ent}(\mathcal{C}^\infty), \quad \hat{\zeta}^\dagger \Psi(z) \equiv \partial_\zeta \Psi(z). \quad (3.18)$$

Their basic properties:

1. Bosonic commutation relation

$$[\widehat{\zeta^*}^\dagger, \hat{\eta}] = \zeta^* \eta, \quad [\hat{\zeta}^\dagger, \widehat{\eta^*}] = \eta^* \zeta. \quad (3.19)$$

2. The exponentials e^{η^*} , $\eta^* \in (\mathcal{C}^{-\infty})$ and e^η , $\eta \in (\mathcal{C}^\infty)$ are the eigenstates of the annihilation operators

$$\widehat{\zeta^*}^\dagger e^\eta = (\zeta^* \eta) e^\eta, \quad \hat{\zeta}^\dagger e^{\eta^*} = (\eta^* \zeta) e^{\eta^*}. \quad (3.20)$$

Creators and annihilators generate strongly continuous abelian operator groups in $\text{Exp}(\mathcal{C}^\infty)$ and $\text{Ent}(\mathcal{C}^\infty)$ parametrized by ζ and ζ^* :

$$e^{\hat{\zeta}} : \text{Exp}(\mathcal{C}^{-\infty}) \rightarrow \text{Exp}(\mathcal{C}^{-\infty}), \quad e^{\hat{\zeta}} \Psi(z^*) = e^{z^* \zeta} \Psi(z^*); \quad (3.21)$$

$$e^{\hat{\zeta}^\dagger} : \text{Ent}(\mathcal{C}^\infty) \rightarrow \text{Ent}(\mathcal{C}^\infty), \quad e^{\hat{\zeta}^\dagger} \Psi(z) = \Psi(z + \zeta); \quad (3.22)$$

$$e^{\widehat{\zeta^*}} : \text{Ent}(\mathcal{C}^\infty) \rightarrow \text{Ent}(\mathcal{C}^\infty), \quad e^{\widehat{\zeta^*}} \Psi(z) = e^{\zeta^* z} \Psi(z); \quad (3.23)$$

$$e^{\widehat{\zeta^*}^\dagger} : \text{Exp}(\mathcal{C}^{-\infty}) \rightarrow \text{Exp}(\mathcal{C}^{-\infty}), \quad e^{\widehat{\zeta^*}^\dagger} \Psi(z^*) = \Psi(z^* + \zeta^*). \quad (3.24)$$

Normal, Weyl, anti-normal quantizations are the operators \hat{M} with *cokernels* $M(\theta^*, \eta)$

$$\hat{M} : \text{Exp}(\mathcal{C}^\infty) \rightarrow \text{Ent}(\mathcal{C}^\infty), \quad M \in \text{Ent}((\mathcal{C}^\infty)^* \times \mathcal{C}^\infty) \quad (3.25)$$

defined by their exponential matrix elements

$$\langle e^{z^*} | \widehat{M}_n | e^w \rangle \equiv \langle M_n(\theta^*, \eta) | \langle e^{z^*} | e^w \rangle \rangle, \quad (3.26)$$

$$\langle e^{z^*} | \widehat{M}_w | e^w \rangle \equiv \langle M_w(\theta^*, \eta) | \langle e^{z^*} | e^{\widehat{\theta} + \widehat{\eta}^*} | e^w \rangle \rangle, \quad (3.27)$$

$$\langle e^{z^*} | \widehat{M}_{an} | e^w \rangle \equiv \langle M_{an}(\theta^*, \eta) | \langle e^{z^*} | e^{\widehat{\theta}^*} e^{\widehat{\eta}} | e^w \rangle \rangle. \quad (3.28)$$

Since

$$\langle e^{z^*} | e^{\widehat{\eta}} e^{\widehat{\theta}^*} | e^w \rangle = \langle e^{\widehat{\eta}^\dagger} e^{z^*} | \widehat{\theta}^* | e^w \rangle \quad (3.29)$$

$$= \langle e^{z^*} \eta e^{z^*} | e^{\theta^* z} e^w \rangle = e^{z^* \eta + \theta^* z} e^{z^* w}, \quad (3.30)$$

one has

$$\langle e^{z^*} | \widehat{M}_n | e^w \rangle = \langle M_n(\theta^*, \eta) | e^{z^* \eta + \theta^* z} e^{z^* w} \rangle = \tilde{M}_n(z^*, w) e^{z^* w} \quad (3.31)$$

where $\tilde{M}_n(z^*, w)$ is the sesqui-linear Fourier transform of $M_n(\theta^*, \eta)$ (see (3.14)). Thus $\tilde{M}_n(z^*, w) e^{z^* w}$ is the normal kernel of \widehat{M}_n .

Since $M_n(\theta^*, \eta)$ is an arbitrary co-kernel, by (3.11), any continuous linear operator Q from $\text{Exp}(\mathcal{C}^\infty)$ to $\text{Ent}(\mathcal{C}^\infty)$ has a unique normal kernel $\tilde{M}_n^Q(z^*, w)$.

By the Taylor expansion centered at origin, the sesqui-entire functionals are uniquely defined by their restrictions to the real diagonal ($z^*, w = z$) so that the *normal symbol* of the operator Q

$$\sigma_n^Q(z^*, z) \equiv e^{-z^* z} \tilde{M}_n^Q(z^*, z) \quad (3.32)$$

exists and defines Q uniquely.

By Baker-Campbell-Hausdorff commutator formula and the canonical commutation relations (3.19),

$$e^{\widehat{\eta}} e^{\widehat{\theta}^*} = e^{\widehat{\eta} + \widehat{\theta}^*} e^{\theta^* \eta / 2}, \quad e^{\widehat{\theta}^*} e^{\widehat{\eta}} = e^{\widehat{\eta} + \widehat{\theta}^*} e^{-\theta^* \eta / 2}. \quad (3.33)$$

Thus any operator Q has Weyl and anti-normal co-kernels M_w^Q and M_{an}^Q . Weyl and anti-normal symbols σ_w^Q and σ_{an}^Q are the normal symbols of \widehat{M}_w^Q and \widehat{M}_{an}^Q .

By (3.33) the symbols of the same operator Q are related via *Weierstrass transform* (cp. AGARWAL-WOLF[1, formulas (5.29), (5.30), (5.31), page 2173] in a finite-dimensional case, DYNIN[5] and [6] in white noise calculus):

$$\sigma_w^Q(z^*, z) = e^{-(1/2)\partial_{z^*} \cdot \partial_z} \sigma_n^Q(z^*, z), \quad (3.34)$$

$$\sigma_{an}^Q(z^*, z) = e^{-N} \sigma_n^Q(z^*, z), \quad (3.35)$$

$$\sigma_w^Q(z^*, z) = e^{(1/2)\partial_{z^*} \cdot \partial_z} \sigma_{an}^Q(z^*, z), \quad (3.36)$$

where the operator $e^{\pm(1/2)\partial_{z^*}\partial_z}$ is the sesqui-linear Fourier transform of the multiplication by $e^{\pm(1/2)\zeta^*\zeta}$, i. e.,

$$\partial_{z^*} \cdot \partial_z \equiv \text{Trace}(\partial_{w^*}\partial_w). \quad (3.37)$$

(Note that $e^{\partial_w^*\partial_w}$ is the Fourier transform equivalent of multiplication by $e^{\theta^*\eta}$ is a continuous operator on $\text{Ent}(\mathcal{C}^{-\infty*} \times \mathcal{C}^{-\infty})$, so that the restrictions to the real diagonal of $e^{\partial_z^*\partial_z}\sigma(z^*, z)$ are well defined.)

Proposition 3.1 *The anti-normal kernel of an operator $Q = \widehat{M}_{an}$*

$$\langle e^{z^*} | Q e^w \rangle = e^{z^*w} \sigma_{an}^Q(w^*, w), \quad (3.38)$$

i.e., Q acts on $\Psi(w^*)$ as the multiplication $\sigma_{an}^Q(w^*, w)\Psi(w^*) \in \text{Ent}(\mathcal{C}^\infty)^* \times \mathcal{C}^\infty$ followed by the orthogonal projection with the kernel $(e^{z^*w})^* = e^{w^*z}$ onto $\text{Ent}(\mathcal{C}^\infty)$.

▷ Since

$$\langle e^{z^*} | e^{\widehat{\theta^*}} e^{\widehat{\eta}} | e^w \rangle = \langle e^{\widehat{\theta^*}} e^{z^*} | e^{\widehat{\eta}} e^w \rangle \quad (3.39)$$

$$= \langle e^{\theta^*z} e^{z^*} | e^{z^*\eta} e^w \rangle = e^{z^*w} e^{\theta^*w + w^*\eta}, \quad (3.40)$$

the anti-normal kernel (3.28)

$$\langle M_{an}(\theta^*, \eta) | e^{z^*w} e^{\theta^*w + w^*\eta} \rangle = e^{z^*w} \widetilde{M}_{an}(w, w^*). \quad (3.41)$$

◁

3.3 Number operator

An operator Q is a *polynomial operator* if any its 1 symbol (and then the other symbols) is a continuous polynomial on $\mathcal{C}^* \times \mathcal{C}$. It belongs to $\text{Exp}(\mathcal{C}^{-\infty})$, and, therefore, their quantizations are tame operators.

The *number operator* $N \equiv \partial_{z^*}\partial_z$ is a polynomial operator with the quadratic symbol $\sigma_n^N = z^*z$. Then, by (3.35) and (3.34) the other symbols

$$\sigma_{an}^N(z^*, z) = z^*z + 1, \quad \sigma_w^N(z^*, z) = z^*z + 1/2. \quad (3.42)$$

The eigenspaces \mathcal{N}_n , $n = 0, 1, 2, \dots$, of N with the corresponding eigenvalue n is the space of continuous homogeneous polynomials of degree n (*n-bosonic states*). In particular, the constant *vacuum state* $\Psi_0 \equiv 1$ corresponds to the eigenvalue $n = 0$.

Gelfand-Schwartz triple (2.21) is the topological orthogonal sum of n -bosons triples

$$\mathcal{N}_n^\infty \subset \mathcal{N}_n^0 \subset \mathcal{N}_n^{-\infty}, \quad n = 0, 1, 2, \dots \quad (3.43)$$

4 Yang-Mills energy-mass operator

4.1 Energy-mass functional

In the first order formalism and the temporal gauge the energy-mass functional (2.8) is (see, e.g. FADDEEV-SLAVNOV [8, Section III.2])

$$H(a, e) = (1/2) \int_{\mathbb{R}^3} d^3x \left((da - [a, a]) \cdot (da - [a, a]) + e \cdot e \right). \quad (4.1)$$

The functional is invariant under the residual gauge transformations subject to the temporal restriction $A_0(t, x^k) = 0$.

Proposition 4.1 *Yang-Mills energy-mass functional of the constrained Cauchy data is gauge equivalent to the non-negative functional*

$$H(a, e) = (1/2) \int_{\mathbb{R}^3} d^3x \left(da \cdot da + [a, a] \cdot [a, a] + e \cdot e \right) \quad (4.2)$$

▷ Let \mathcal{G}^0 denote the completion of \mathcal{G}^∞ with respect to the natural \mathcal{L}^2 -metric on the transformations of \mathcal{S}^0 . Then, by DELL'ANTONIO-ZWANZIGER [4, Proposition 4],

1. The gauge action of \mathcal{G}^∞ on $\mathcal{A}^\infty \times \mathcal{S}^\infty$ has a unique extension to the continuous action of \mathcal{G}^0 on

$$\mathcal{C}^0 \equiv \mathcal{A}^0 \times \mathcal{S}^0. \quad (4.3)$$

2. The gauge orbits of this action are closures of \mathcal{G}^∞ -orbits.
3. On the euclidean orbit of every a the Hilbert \mathcal{L}^2 -norm $\|a^g\|$ attains the absolute minimum at some gauge equivalent connection $\check{a} \in \mathcal{A}^0$.
4. Minimizing connections \check{a} are weakly divergence free: $\partial^k \check{a}_k = 0$.

Thus the cubic term in $(da - [a, a]) \cdot (da - [a, a])$ vanishes on the minimizing connections. ◁

4.2 Yang-Mills energy-mass operator and its symbols

Let the *quantum Yang-Mills energy-mass operator* $H : (\mathcal{C})^\infty \rightarrow \text{Exp}(\mathcal{C}^{-\infty})$ be the anti-normal quantization of the energy-mass functional H :

$$\sigma_{an}^H(a, e) \equiv H(a, e) = H(z^*, z), \quad \zeta = a + ie, \quad \zeta^* = a^T - ie^T, \quad (4.4)$$

i.e. H is the *anti-normal symbol* of H .

The *expectation functional* on non-zero $(\mathcal{C})^\infty \in (\mathcal{C})^\infty$ of a polynomial operator Q is

$$\langle Q \rangle(\Psi) \equiv \Psi^* Q \Psi / \Psi^* \Psi. \quad (4.5)$$

Proposition 4.2 *There exists a constant scalar field C on \mathbb{R}^3 such that the expectation functional*

$$\langle H \rangle \geq \langle N \rangle + \langle C \rangle, \quad (4.6)$$

where N is the number operator (3.42).

▷

(A) Let M be the operator with the non-negative anti-normal symbol

$$\Omega_{an}^M(z^*, z) \equiv \int_{\mathbb{R}^3} d^3x ([a, a] \cdot [a, a] + e \cdot e). \quad (4.7)$$

Then, by Propositions 4.1 and 3.1,

$$\langle H \rangle \geq \langle M \rangle. \quad (4.8)$$

(B) Let b_i be a basis for $\text{Ad}(\mathbb{G})$ with $b_i \cdot b_j = \delta_{ij}$. Then the structure constants $c_{ij}^k = [b_i, b_j] \cdot b_k$ are anti-symmetric under interchanges of i, j, k . Thus if $a = a^i b_i \in \text{Ad}(\mathbb{G})$ then

$$a \cdot a = \text{Trace}(a^t a) = -a^i c_{ij}^k a^l c_{kj}^l = a^i c_{ij}^k a^l c_{lj}^k, \quad (4.9)$$

so that

$$[a, a] \cdot [a, a] = a^i a^j a^l a^m [b_i, b_j] \cdot [b_l, b_m] \quad (4.10)$$

$$= a^i a^j a^l a^m c_{ij}^k c_{lj}^k = \sum_k (a^i a^j c_{ij}^k)^2. \quad (4.11)$$

Since $z = (1/\sqrt{2})(a + ie^o)$, the operator $\partial_z^* \partial_z = \partial_a^2 + \partial_{e^o}^2$, so that

$$e^{\partial_z^* \partial_z / 2}([a, a] \cdot [a, a]) \stackrel{(4.9)}{=} [a, a] \cdot [a, a] / 2 + a \cdot a. \quad (4.12)$$

Then there is a constant scalar field C such that the Weyl symbol of the operator M

$$\sigma_w^M(a, e) \stackrel{(3.36), (4.12)}{=} \int_{\mathbb{R}^3} d^3x ([a, a] \cdot [a, a] / 2 + a \cdot a + e \cdot e) + 1/2 + C. \quad (4.13)$$

(C) The Weyl quantization of $[a, a] \cdot [a, a]$ is the operator of multiplication with $[a, a] \cdot [a, a] \geq 0$ in the " (a, e) -representation" of the canonical commutation relations (cp. AGARWAL-WOLF [1, Section VII, page 2177]). In particular, its expectation functional is non-negative.

(D) By (3.42), $\int_{\mathbb{R}^3} d^3x (a \cdot a + e \cdot e) + 1/2$ is the anti-normal symbol of the number operator N .

Thus

$$\langle M \rangle \geq \langle N \rangle + \langle C \rangle. \quad (4.14)$$

The proposition follows from the inequalities (4.8) and (4.14). \triangleleft

4.3 Bosonic spectrum of Yang-Mills energy-mass operator

Operator H is a non-negative polynomial symmetric operator. It has a unique Friedrichs extension to an unbounded self-adjoint operator on the Hilbert space $\mathcal{B}(\mathcal{C}^0)$. Proposition 4.2 implies via the variational mini-max principle (see, e.g. BEREZIN-SHUBIN [2, Appendix 2, Proposition 3.2]) that its spectrum is degenerate along with the spectrum of the number operator.

To remove the degeneracy, consider the n -particle spaces \mathcal{N}_n^∞ as elementary bosons of spin n . Then define the *bosonic spectrum* of H as the non-decreasing sequence of its *spectral values*

$$\lambda_n(H) \equiv \inf\{\langle H \rangle(\Psi), \Psi \in \mathcal{N}_n^\infty\}. \quad (4.15)$$

Proposition 4.2 implies the main

Theorem 4.1 *The bosonic spectrum of Yang-Mills energy-mass operator H is infinite and discrete, i.e. each $\lambda_n(H)$ has a finite bosonic multiplicity.*

The bosonic spectral values grow at least in the arithmetical progression:

$$\lambda_n(H) \geq n + \text{constant}, \quad n = 0, 1, 2, \dots \quad (4.16)$$

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